Rectangular Fin Thermal Performance as a Function of Modified Biot Number and Area Ratio; Novel Approach

Al-Fahed S. F. and Alasfour F. N.

Abstract— A novel analytical study of 1-D, heat conduction equation in a longitudinal fin of a uniform thickness is presented. A new perspective in fin modeling have demonstrated that three thermal performance indicators; efficiency, effectiveness and performance ratio can be presented only in terms of two dimensionless groups; modified Biot number(hL/k) and fin area ratio (A/Ac). A suggested fin design charts are presented in terms of the three thermal performance indicators. Optimum fin design charts been presented as a function of only modified Biot number and area ratio. Results showed that such design charts are more practical and form an alternative route to interpret fin thermal behavior and performance.

Keywords— Biot, Effectiveness, Efficiency, Fin, Performance, Rectangular, Thermal

1 INTRODUCTION

Fins are used to enhance heat transfer rate from a primary surface by increasing the effective surface area. Three major parameters effect heat transfer rate; surface area, temperature gradient, and thermal conductivity. Fins are extensively used to enhance and augment heat transfer from surfaces in heat exchangers; thus making heat exchangers compact and more efficient. Such heat exchangers are widely used in domestic and industrial equipment, examples are air conditioning, heating, and dehumidification, refrigeration systems, etc. Hence, assessing the performance thermal indicators of a given fin has a significant importance in optimization, and producing design charts.

Although numerous mathematical models and heat transfer analyses exist in the literature, it is noted that there is a vast need for further improvements in designing efficient and effective fins. The conventional performance of a given fin is usually presented via three common thermal indicators; (1) fin efficiency, (2) effectiveness, and (3) performance ratio.

Fin efficiency is expressed as the ratio of fin actual heat transfer rate to the heat transfer rate when the whole fin is at base temperature. Note that when the whole fin at a hypothetical base temperature, heat transfer is maximum; so thermal conductivity is infinite. The disadvantage of fin efficiency indicator is that the real performance is compared to a non-existing fin with infinitely high thermal conductance, so fin efficiency is an idealization and physically has no meaning [1]. In the same manner Heggs and Ooi [2] "stated that fin efficiency does not reflect the physics of heat flow through fin geometry, it was shown to be antithesis of performance indicator".

Fin effectiveness is the ratio of actual heat transfer rate through the fin to the one from the primary surface of the detached fin, in another word, effectiveness is defined as rate of fin heat transfer to the rate of heat transfer would exist without fin [3]. It seems that effectiveness should be as high as possible. For practical design, if $\xi \leq 2$, fin will not be appropriate [4]. There are two main factors that increase fin effectiveness; (1) high thermal conductivity, (2) high value of area ratio(AR).

Finally the fin performance ratio is defined as the ratio of actual heat transfer rate to the heat transfer rate through a similar fin of infinite length. Fin performance was proposed by Heggs [5], he believed that performance is better thermal indicator than fin efficiency. Such three indicators are widely used in evaluating fin performance.

One of the earliest works on fins was performed by Harper and Brown [6], they presented equations for fin efficiency for rectangular and annular type with uniform cross sectional area that are used in aircraft engines. Later Murrary [7] analyzed temperature distribution and effectiveness of annular fins with constant thickness. An extensive review on fins and their use is presented by Kern and Kraus [8]. Incropera et al. [3] devoted a considerable effort to present analytical solutions and design charts for temperature distribution, heat flow rate, efficiency and fin effectiveness for a variety of fin shapes and configurations. Laor and Calman [9] presented a study on the performance and optimum size of a longitudinal and annular fins. Bejan [10] , based on fin effectiveness and Sinder and Kraus[11],

- *Sami F. Al-Fahed is a Professor in Mechanical Engineering Department at Kuwait University. Kuwait. Email: mmaq65@gmail.com*
- *Fuad N. Alasfour is a Professor in Mechanical Engineering Department at Kuwait University. Kuwait. Email: fuadalasfour@gmail.com*

based on the largest transverse temperature variation , concluded that the one-dimensional model for fins is valid for $Bi <$ < 1. Later Aziz and Lunardini [12] , in their study of fins, demonstrated that the error in the dimensionless heat flow rate calculated from the one dimensional model as compared to the two dimensional model, for rectangular fins, is approximately 1% for $Bi = 0.1$. The concept of performance ratio was first introduced by Heggs [5]. Then, Heggs [2] presented design charts for the performance ratio of radial rectangular fins.

In most of the previous researches one can see that the three thermal performance indicators depend mainly on the profile area of the fin, the parameter m, and Biot number which is based on fin thickness [3]. In the present research an attempt is made to investigate an alternative analytical route to relate the three thermal performance indicators. The idea is to introduce a modified Biot number based on the fin length rather than fin thickness and area ratio. The suggested modified Biot number which is based on fin length can be more descriptive to heat transfer process through a one-dimensional fin case.

The possibility of having fin thermal performance indicators such that they depend on area ratio and modified Biot number based on fin length rather than on fin thickness were presented in some of the previous works. Several attempts been experienced through numerous publications to formulate fin thermal performance, through (design charts, by introducing a non-dimensional parameters). Such that design charts can be utilized in terms of those parameters. Lane and Heggs [3] performed a study on different fin configurations for temperature, heat transfer rate, and performance ratio with various fin profiles: dovetail, trapezoidal, and tapered. Their heat flow rate and performance ratio relations, for the rectangular fin have demonstrated a dependence on the term of area ratio AR. On the other hand, the heat transfer rate relation for the dovetail fin showed a dependence on$(AR)^{1/2}$.

Aziz [14] and Aziz and Addendum[15] presented an exact solution for the rectangular fins with insulated tip and constant thermal conductivity, the solutions for both temperature distribution and heat flow rate demonstrated the dependence on modified Bi (hL/k) and $(\mathrm{hp}L^2)/kA_c$ basically Bi*AR. Also Aziz and Beers-Green[16] investigated the performance and optimum design of longitudinal rectangular fins. In their solution for temperature distribution, heat transfer rate, and fin efficiency, they presented Biot number based on fin length ($Bi = hL/k$) and a term $(\mathrm{hp}L^2)/kA_c$ which is basically $Bi * AR$. Arslanturk [17] in an attempt to optimize the design of annular fins with uniform cross section has shown in his relation for the dimensionless heat transfer rate $Bi^{1/2}$ dependence, where Bi is based on inner radius of the fin. Harley and Moitsheki [18] noted in their work on rectangular fins that a thermo-geometric parameters be introduced to investigate the influence on temperature distribution, hence on fin thermal performance indicators.

Although fin thermal performance indicators such as fin efficiency is widely used, it does not seem to be suitable for optimum design [17]. So, the objective of the present work is to determine the dependence of the three thermal performance indicators on two practical parameters; fin area ratio and modified Biot number based on fin length. In addition this work aims to produce fin design charts for the purpose of optimizing fin performance indicators.

2 Mathematical Modeling

The general fin temperature distribution across variable cross section with heat generation can be presented as

$$\frac{\partial^2 T}{\partial x^2} + \frac{1}{A_c}\frac{dA_c}{dx}\frac{dT}{dx} - \frac{h(T-T_\infty)}{k\,A_c}\frac{dA}{dx} + \frac{\dot{q}}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1}$$

In this research the following assumptions will be used to define the limits of a constant cross section 1-D fin model.

1. Steady state condition where temperature distribution and heat transfer rate remain constant with time.
2. The variation of temperature only in the lateral direction, $T = T(x)$.
3. Constant and uniform convective heat transfer coefficient (h) over the entire surface including fin tip.
4. Constant thermal conductivity (homogenous and isotropic).
5. Bi is calculated based on fin length as characteristic length, $Bi = hL/k$.
6. Uniform temperature at the fin base.
7. There is no heat generation within the fin and no radiation effect.

The governing differential equation for the temperature distribution along the fin can be reduced to a second order differential equation

$$\frac{d^2\theta}{dx^2} - m^2\,\theta = 0 \tag{2}$$

where, $\theta = T - T_\infty$

θ represents excess temperature and the constant m^2 *is equal to* hp/kA_c .

The solution of the above ordinary differential equation can be presented in a normalized temperature profile

$$\frac{\theta}{\theta_b} = \frac{[\cosh m\,(L-x) + h/mk\,\sinh m\,(L-x)\,]}{[\,\cosh mL + h/mk\,\sinh mL]} \tag{3}$$

where, boundary conditions are at fin base:

$x = 0, \theta = T_b - T_\infty = \theta_b$

at fin tip: $x = L, -k\,A_c\,\frac{d\theta}{dx}\Big|_{x=L} = hA_c\theta_L$

The boundary condition at the tip represents the case of convective heat transfer at the fin tip, and $\theta_L = T_L - T_\infty$

Hence, the total heat transfer rate by the fin can be determined as follows:

$$q = q_b \tag{4}$$

where, $q_b = -kA_c\frac{d\theta}{dx}\Big|_{x=0}$

hence,

$$q = [(\,hpkA_c)^{1/2}\,\theta_b]\left[\frac{\tanh mL + h/mk}{1 + h/mk\tanh mL}\right] \tag{5}$$

the conventional fin efficiency is expressed as :

$$\eta = \frac{q}{q_{\max}} \tag{6}$$

where, $q_{max} = h\,A\,\theta_b$

thus,

$$\eta = \left[\frac{(\,hpkA_c)^{1/2}}{hA}\right]\left[\frac{\tanh mL + h/mk}{1 + (h/mk)\tanh \mathrm{mL}}\right] \tag{7}$$

Fin effectiveness, ξ , is defined as :

$$\xi = \frac{q}{hA_c\theta_b}$$

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thus,

$$\xi = \left[\frac{pk}{hA_c}\right]^{1/2} \left[\frac{\tanh mL + h/mk}{1 + (h/mk)\tanh mL}\right] \quad (8)$$

The fin performance ratio is expressed as

$$PR = \frac{q}{q_{L\to\infty}} \quad (9)$$

hence,

$$PR = \left[\frac{\tanh mL + h/mk}{1 + h/mk \tanh mL}\right] \quad (10)$$

It is clear that the four conventional non dimensional groups: $[mL, h/(mk), (hpkA_c)^{1/2}/hA \text{ and } (pk/hA_c)^{1/2}]$ influence fin temperature distribution and the three performance indicators. $[\frac{\theta}{\theta_b}, \eta, \xi \text{ and } PR]$ will be expressed as functions of AR and Bi.

In the following analytical section an attempt will be made to demonstrate that the four conventional groups can be expressed as a function of modified Bi and AR. Where, AR is fin area ratio; $AR = A/A_c$ and modified Bi number (hL/k) is a function of characteristic fin length which represents the direction of conduction heat flow.

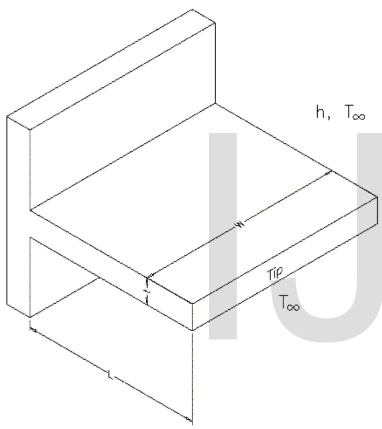

Figure 1 Schematic diagram of the rectangular fin

By considering a 1-D fin as in Fig. 1, a non-dimensional parameters a_1 and a_2 can be introduced as such:

$$a_1 = Bi^{1/2}(AR-1)^{1/2} \quad (11)$$

$$a_2 = Bi^{1/2}\left[\frac{1}{AR-1}\right]^{1/2} \quad (12)$$

Such that the four conventional groups can be presented, after some manipulations, as the following (Appendix A).

$$mL = Bi^{1/2}\,[AR - 1]^{1/2} \quad (13)$$

$$h/(mk) = Bi^{1/2}\left[\frac{1}{AR-1}\right]^{1/2} \quad (14)$$

$$\frac{(hpkA_c)^{1/2}}{hA} = \left[\frac{1}{Bi^{1/2}AR}\right][AR-1]^{1/2} \quad (15)$$

$$\left[\frac{pk}{hA_c}\right]^{1/2} = \left[\frac{1}{Bi^{1/2}}\right]\left[\frac{A-A_c}{A_c}\right]^{1/2} = \left[\frac{1}{Bi^{1/2}}\right][AR-1]^{1/2} \quad (16)$$

Thus, fin temperature distribution can be presented as:

$$\frac{\theta}{\theta_b} = \left[\frac{\cosh a_1\left(1-\left(\frac{x}{L}\right)\right) + a_2 \sinh a_1\left(1-\left(\frac{x}{L}\right)\right)}{\cosh a_1 + a_2 \sinh a_1}\right] \quad (17)$$

and fin efficiency,

$$\eta = \frac{[a_1/(Bi\,AR)][\tanh a_1 + a_2]}{1 + a_2 \tanh a_1} \quad (18)$$

where, fin effectiveness can be expressed as:

$$\xi = \frac{[a_1/Bi][\tanh a_1 + a_2]}{1 + a_2 \tanh a_1} \quad (19)$$

and fin performance ratio:

$$PR = \frac{\tanh a_1 + a_2}{1 + a_2 \tanh a_1} \quad (20)$$

By comparing equations 18, 19, and 20, one can obtain the following relations between fin effectiveness and performance ratio:

$$\xi = AR * \eta \quad (21)$$

$$PR = [(Bi\,AR)/a_1]\ \eta = (Bi/a_1)\ \xi \quad (22)$$

and the ratio of maximum heat transfer rate by the fin to the base heat transfer rate as:

$$q_{max}/q_b = (hA\,\theta_b)/(hA_c\,\theta_b) = A/A_c = AR \qquad AR > 1 \quad (23)$$

The maximum effectiveness ξ_{max} can be obtained from Eq. 21 by setting η equal to one, hence

$$\xi_{max} = AR \quad (24)$$

Then the influence of the area ratio on the thermal performance indicators can be demonstrated as the following:

$$AR = 1 + L/(t/2) + L/(w/2) \quad (25)$$

The first term of Eq. 25 represents the primary area of the detached fin. The second term represents the fin aspect ratio, and the third term represents the fin length to half width ratio. When L is equal to zero it means fin is detached or there is no fin. The second term, $L/(t/2)$, will have the major effect, while the last term $L/(w/2)$, except for the cases when w is less than L, will have the least influence on performance indicators. Thus the analytical solution leads to area ratio as a primary dimensionless group in the thermal performance indicators.

3 Results and Discussion

Based on the previous analytical solution, the temperature distribution, fin efficiency, effectiveness and performance ratio were determined and plotted versus different area ratios and modified Biot numbers. The dimensionless temperature distribution along the fin for different area ratios and modified Biot numbers are shown in Figures 2-4. It can be seen that in the case of small modified Biot number (0.001) the temperature of the fin is nearly equal to the base temperature for the three area ratios 5, 10, and 20; hence convective heat transfer has a minimum effect. As modified Biot number increases, the temperature shows a drop along the fin's length due to the increase in conduction resistance, one can see that for the case AR =20, which

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simulates long fin case, and Bi=1 drop is nearly 56%. Also it is noticed that the effect of AR on temperature distribution, results show that as AR increases, the drop in temperature along the fin is pronounced at higher values of Bi number.

The fin efficiency versus AR for different values of Bi is shown in Figure 5. Results showed that efficiency decreases with the increase in AR, and it is more noticeable at higher Bi number values. On the other hand, for small Bi value, the efficiency is nearly constant regardless of AR. For the case of high Bi (Bi = 10), the efficiency is very low for most values of area ratios. Result predict that at $Bi = 1$; fin conduction is equal to convection, shows poor values in thermal efficiency, about 30 % at $AR = 10$, then efficiency reduces to values of 12 % as AR increases. Figure 5 also showed the behavior of efficiency under the assumption of a theoretical value of $Bi = 10$, where the mode of convection heat transfer is predominant compared to conduction heat transfer mode, the fin thermal efficiency showed a low (uneffected) value of about 5%.

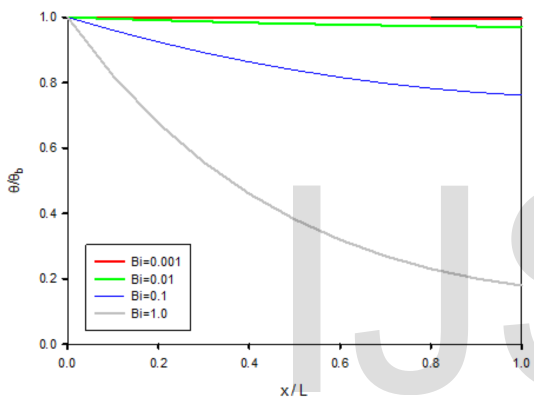

Figure 2 Temperature distribution along constant cross sectional rectangular fin at AR = 5.

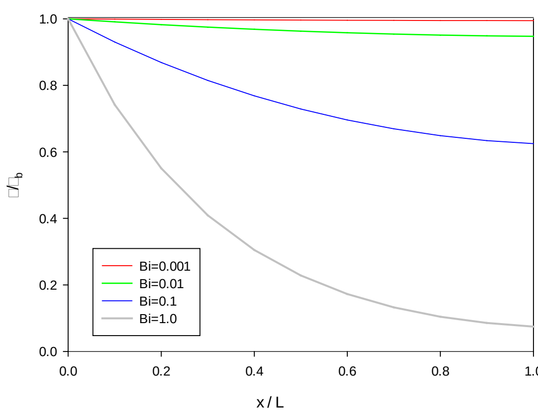

Figure 3 Temperature distribution along constant cross sectional rectangular fin at AR = 10.

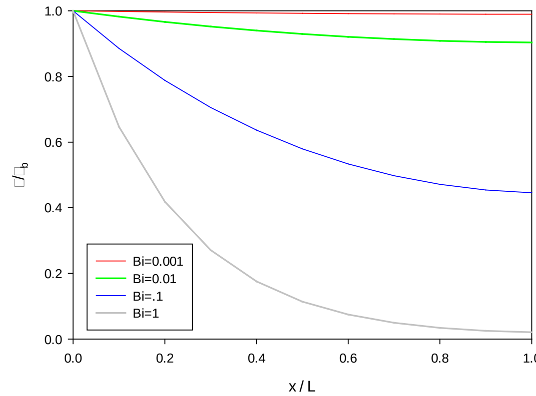

Figure 4 Temperature distribution along constant cross sectional rectangular fin at AR = 20.

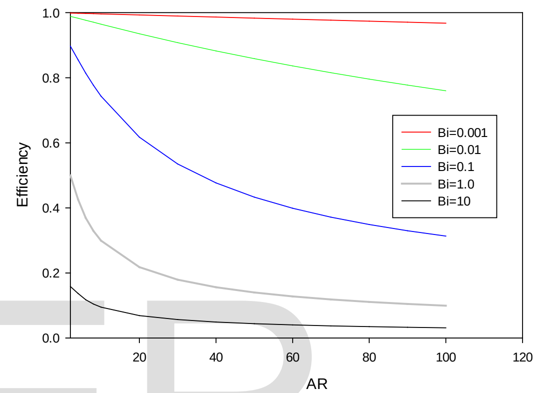

Figure 5 The effect of AR on rectangular fin efficiency at different modified Bi numbers.

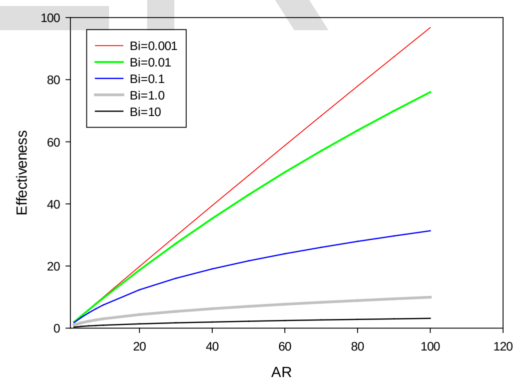

Figure 6 The effect of AR on rectangular fin effectiveness at different values of modified Bi numbers.

The fin effectiveness versus area ratio for different modified Bi numbers is shown in Figure 6. For small Biot numbers (0.001) the effectiveness is linearly dependent on AR. As Biot number increases, the effectiveness deviates from linear behavior, and becomes less effect as AR increases. For high theoretical values of Biot numbers such as 10, fin effectiveness is minimal and nearly constant regardless of AR.

The values of fin performance ratio as a function of AR for different Bi numbers are depicted in Figure 7. It can be seen that for Bi number equal to 0.001, the performance ratio is very low for all area ratios. As the Bi number reaches 0.01, the area ratio influence on the performance ratio is obvious. As the area ratio

increase, performance rato increases. For large theoretical values of $Bi(Bi = 10)$, the performance ratio is equal to one.
Design charts for efficiency, effectiveness and performance ratio are presented in Figures 8-10. Figure 8 shows the first set of fin design charts. Efficiency is presented as a function of modified Bi number and area ratio. Results reveal that at constant cross sectional area, short fins (lower value of AR) provides higher value of fin efficiency, another observation in figure 8, at constant Bi number (hL/k), efficiency is inversely proportional with AR. Results in Figure 8 leads to short fins with low value of modified Boit number provide an elevated values in efficiency and as modified Biot number increases efficiency getting less.
Figure 8 also shows that efficiency converges to a value of one for small values of Bi numbers for all area ratios. All efficiency curves will converge for large Bi numbers to nearly 5%. In general, the efficiency decreases with an increase in Bi number, however, large AR will experience faster drop in efficiency as Bi increases. For fin between $Bi\ equal\ to$ 0.01 and 1, the choice of area ratio plays an important role on fin efficiency.

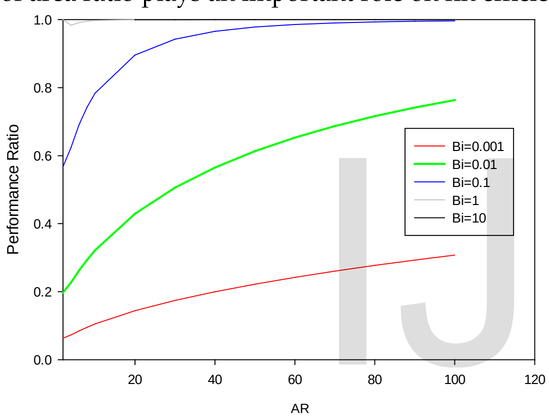

Figure 7 The effect of AR on performance ratio of rectangular fin at different modified Bi numbers.

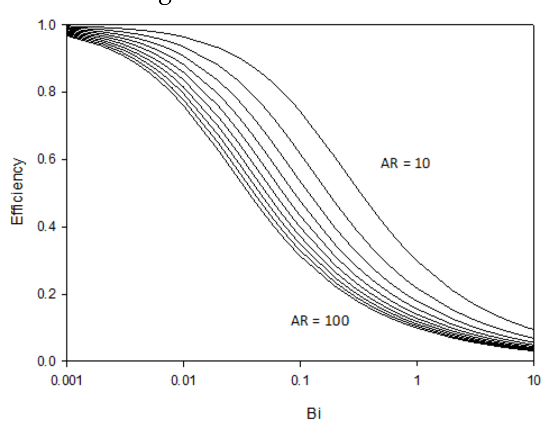

Figure 8 Efficiency design chart for rectangular fin as a function of Bi number at different AR.

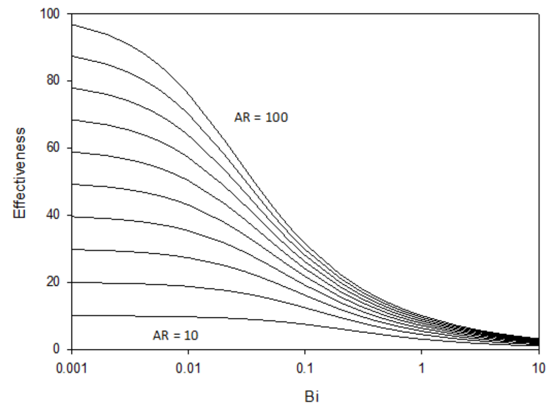

Figure 9 Effectiveness design chart for rectangular fin as a function of Bi number at different AR.

In Figure 9, fin effectiveness as a function of Bi for different AR is shown. For a given area ratio, the effectiveness decreases with an increase in Bi number, and reaches asymptotic value for $Bi \geq 10$. Large area ratios and small Biot numbers demonstrate high value of effectiveness, and as area ratio is reduced, the fin becomes less effective. Figure 9 also demonstrates the fact when the whole fin will be at the base temperature; Bi is very small; hence the conductive resistance through the fin is in its minimal value, equation 21 predicted effectiveness is proportional to η where proportionality factor is AR, hence such thermal behavior of η and ξ goes back to the definition of modified Biot number(hL/k). As the Bi number increases the effectiveness deviates for large area ratios values because the efficiency influences the effectiveness. Because of such behavior, both efficiency and effectiveness behave in a similar manner with respect to Bi, except for the fact they behave in opposite directions with respect to AR. For a given Bi, higher value of AR reduces efficiency at the same time increases effectiveness. Thus there should be some trade between efficiency and effectiveness.

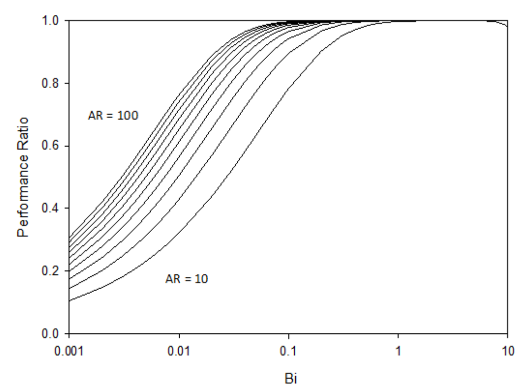

Figure 10 Performance design chart for rectangular fin as a function of Bi number at different AR.

The performance ratio as a function of Biot number and area ratio is shown as design chart in Figure 10. Results show that as the Bi number increases, the PR increases first slowly then sharply in the range $0.01 \leq Bi \leq 0.1$, then it reaches asymptotic value equal to one. High area ratios associated with high

PR as at constant value of Bi, The PR reaches one at smaller values of Bi numbers. Performance ratio and efficiency behave in opposite directions with respect to both Bi and AR, an increase in Bi will reduce efficiency and increases PR. For a specific value of AR, an increase in Bi will decrease efficiency and in contrast increases PR. For a given Bi, an increase in AR will increase PR but reduces efficiency. Heggs [9, 10] believes that how well a fin performs is important rather than how efficient it is.

4 Conclusion

A novel mathematical model for 1-D longitudinal fin with uniform thickness is derived and presented. An alternative analytical route been applied to relate the three fin thermal performance indicators; efficiency, effectiveness and performance ratio. The uniqueness of the analytical model is that the three thermal indicators been derived in terms of modified Biot number and area ratio.

To the best of our knowledge, it is the first time that an optimum fin design charts been presented as a function of only modified Biot number and area ratio. Novelty results showed that such design charts can form more thermal practical approach and present an alternative route to interpret fin thermal behavior and thermal performance.

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Nomenclature

A fin total surface area, m^2
A_c fin cross-sectional area, m^2
a_1, a_2 non dimensional constants
AR fin area ratio (A/A_c)
Bi modified Biot number, Bi=hL/k
h convection heat transfer coefficient W/m2-K
k thermal conductivity of fin, W/m-K
L fin length, m
m is a constant, $m = hp/kA_c$
p perimeter of the fin
PR performance ratio
q heat transfer rate,
$\dot{q}$ heat generation, W/m3
T temperature, K
T_∞ surrounding fluid temperature, K
t fin thickness, m
w fin width, m
x distance along the fin, m

Greek symbols
α thermal diffusivity, m2/s
ϴ excess temperature, K
η fin efficiency
ξ fin effectiveness

Subscripts
b fin base
c cross section
max maximum

Appendix

$\boldsymbol{mL}$ parameter,

$mL = [(hp)/(kA_c)]^{1/2}L = [(hpL^2)/(kA_c)]^{1/2} = [(hL/k)(pL/A_c)]^{1/2}$

$mL = Bi^{1/2}[(pL)/A_c]^{1/2} = Bi^{1/2}[(A - A_c)/A_c]^{1/2}$

hence, $mL = Bi^{1/2}[A/A_c - 1]^{1/2} = Bi^{1/2}[AR - 1]^{1/2}$ (13)

$\boldsymbol{h/mk}$ parameter,

$h/(mk) = (h/k)[(kA_c)/(hp)]^{1/2} = [(hA_c)/(kp)]^{1/2} = [(hL)/k]^{1/2} * [A_c/(pL)]^{1/2}$

hence,

$h/(mk) = Bi^{1/2}[A_c/(A - A_c)]^{1/2} = Bi^{1/2}[1/(AR-1)]^{1/2}$ (14)

$(\boldsymbol{hpkA_c})^{\mathbf{1/2}}/(\boldsymbol{hA})$, parameter,

$(hpkA_c)^{1/2}/(hA) = (1/A)[(pkA_c)/h]^{1/2} = (1/A)[(pkLA_c)/(hL)]^{1/2}$

thus, $(hpkA_c)^{1/2}/(hA) = (1/A)[k/(hL)]^{1/2}(pLA_c)^{1/2}$

$(hpkA_c)^{1/2}/(hA) = [1/Bi^{1/2}][((A - A_c)/A)(A_c/A)]^{1/2}$

hence,

$(hpkA_c)^{1/2}/(hA) = [1/(Bi^{1/2}AR)][AR - 1]^{1/2}$ (15)

$[(\boldsymbol{pk})/(\boldsymbol{hA_c})]^{\mathbf{1/2}}$ parameter,

$[(pk)/(hA_c)]^{1/2} = [(pLk)/(A_chL)]^{1/2}$

$= [k/(hL)]^{1/2}[(pL)/A_c]^{1/2}$

$\left[\frac{pk}{hA_c}\right]^{\frac{1}{2}} = \left[\frac{1}{Bi^{\frac{1}{2}}}\right]\left[\frac{A-A_c}{A_c}\right]^{\frac{1}{2}} = [1/Bi^{1/2}][AR-1]^{1/2}$ (16)